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Periodic potentials and PT symmetry

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Abstract

We enlarge the limited class of analytically solvable one-dimensional periodic potentials with a finite number of band gaps by considering the complex, PT-symmetric generalized associated Lamé (GAL) potential with four parameters and applying supersymmetry. We obtain three duality relations which relate the band-edge eigenstates of the twenty four GAL potentials obtained by permutation of the four parameters.

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1. Introduction

In the past few years, Bender and others [1] have studied several complex potentials with PT-symmetry. They have shown that the energy eigenvalues are real when PT-symmetry is unbroken, whereas the eigenvalues occur in complex conjugate pairs when PT-symmetry is spontaneously broken. There have also been a few papers discussing periodic potentials with PT-symmetry [2–4].

A well-studied periodic problem is the Lamé potential [5–7]

$$V_L(x) = a(a+1)m\,\mathrm{sn}^2(x,m). \tag{1}$$

Here, $\mathrm{sn}(x,m)$ and its two companions $\mathrm{cn}(x,m)$, $\mathrm{dn}(x,m)$ are Jacobi elliptic functions with elliptic modulus parameter $m(0 \leqslant m \leqslant 1)$. They are doubly periodic functions with periods $[4K(m), \mathrm{i}2K'(m)]$, $[4K(m), 2K(m) + \mathrm{i}2K'(m)]$, $[2K(m), \mathrm{i}4K'(m)]$, respectively [9], where $K(m) \equiv \int_0^{\pi/2} \mathrm{d}\theta[1 - m\sin^2\theta]^{-1/2}$ denotes the complete elliptic integral of the first kind and $K'(m) \equiv K(1-m)$. If the parameter a in the Lamé potential is a positive integer, it is well known that there are a band gaps [6] and hence $2a+1$ band edges, whose energies are analytically known (in principle). A few years ago, we extended these results and obtained the band edges of the associated Lamé (AL) potential [8]

$$\begin{aligned} V_{\mathrm{AL}}(x) &= a(a+1)m\,\mathrm{sn}^2(x,m) + b(b+1)m\,\mathrm{sn}^2(x+K(m),m) \\ &= a(a+1)m\,\mathrm{sn}^2(x,m) + b(b+1)m\frac{\mathrm{cn}^2(x,m)}{\mathrm{dn}^2(x,m)}, \end{aligned} \tag{2}$$

which involves two parameters a, b. It was shown that the AL potentials with integer values of a, b are periodic potentials with a finite number of band gaps which are analytically known [10, 11]. We also showed that applying supersymmetry to the Lamé or the AL potentials yields additional novel potentials with a finite number of band gaps [3, 11]. Besides, quasi-exactly solvable (QES) elliptic potentials with an infinite number of band gaps have also been discussed [12].

The purpose of this short paper is to show that the list of periodic potentials with a finite number of band gaps can be substantially enlarged if we look at complex, PT-invariant periodic potentials. We consider the four-parameter family of generalized associated Lamé (GAL) potentials

$$\begin{aligned} V_{\mathrm{GAL}}(x) &= a(a+1)m\,\mathrm{sn}^2(y,m) + b(b+1)m\,\mathrm{sn}^2(y+K(m),m) \\ &\quad + f(f+1)m\,\mathrm{sn}^2(y+K(m)+\mathrm{i}K'(m),m) + g(g+1)m\,\mathrm{sn}^2(y+\mathrm{i}K'(m),m) \\ &= a(a+1)m\,\mathrm{sn}^2(y,m) + b(b+1)m\frac{\mathrm{cn}^2(y,m)}{\mathrm{dn}^2(y,m)} \\ &\quad + f(f+1)\frac{\mathrm{dn}^2(y,m)}{\mathrm{cn}^2(y,m)} + g(g+1)\frac{1}{\mathrm{sn}^2(y,m)} \\ &\equiv [a,b,f,g], \end{aligned} \tag{3}$$

where $y = x - K(m)/2 - \mathrm{i}K'(m)/2$. Using the shifted variable y is a simple way to ensure that the potential has no singularities on the real x-axis coming from the zeros of the Jacobi elliptic functions $\mathrm{sn}(x,m)$ and $\mathrm{cn}(x,m)$. Note that we shall frequently use the notation $[a,b,f,g]$ to denote the GAL potential $V(x)$. In this notation, ordinary Lamé potentials are denoted by $[a,0,0,0]$ and AL potentials are denoted by $[a,b,0,0]$.

Even though the GAL potential $[a,b,f,g]$ is complex, it has real eigenvalues by virtue of unbroken supersymmetry. Remarkably, as described below, it turns out that GAL potentials have a finite number of band gaps when a, b, f, g are all integers.

Given any complex, PT-symmetric potential with a finite number of band gaps, supersymmetry allows us to readily obtain several more potentials with the same energy levels and consequently the same finite number of band gaps. Normally, in supersymmetric quantum mechanics [13], given a potential $V_-(x)$, the ground-state wavefunction $\psi_0(x)$ is used to construct the superpotential $W(x) = -\psi_0'(x)/\psi_0(x)$, which then yields the supersymmetric (SUSY) partner potential $V_+(x) = W^2 + W'$. If one uses any excited-state wavefunction $\psi(x)$ of $V_-(x)$ to construct a superpotential $W(x)$, then the original potential $V_-(x)$ is recovered correctly (by construction), but the corresponding partner potential $V_+(x)$ turns out to be singular on the real x-axis due to the zeros of the excited-state wavefunction $\psi(x)$. However, as has been noted recently [14], if we consider PT-symmetric complex potentials, then the singularity is not on the real axis. In this way, by starting from the analytically solvable Lamé, AL and GAL potentials and using the excited-state band edges, we discover a wide range of new, analytically solvable, complex PT-invariant periodic potentials with a finite number of band gaps.

2. Generalized associated Lamé (GAL) potentials

Our main task is to solve the Schrödinger equation for the GAL potential (3) given by ($\hbar = 2m = 1$)

$$-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\psi(x) + V_{\mathrm{GAL}}(x)\psi(x) = E\psi(x). \tag{4}$$

More completely, displaying the parameters for the GAL potential $[a, b, f, g]$, the energy is $E(a, b, f, g; m)$ and the wavefunction is $\psi(x, a, b, f, g; m)$. Let us point out several important properties of the GAL potential.

2.1. Finite number of band gaps

GAL potentials are known in the mathematics literature as Treibich–Verdier potentials [15] even though they were discussed by Darboux [16], Hermite and Sparre [17] more than a hundred years ago. Several people have shown that when a, b, f, g are integers, then potential (3) has a finite number p of band gaps given by [18]

$$p = \tfrac{1}{2}\max\big[2\max[a, b, f, g], 1 + N - \{1 + (-1)^N\}\big\{\min[a, b, f, g] + \tfrac{1}{2}\big\}\big], \quad (5)$$

where $N \equiv a + b + f + g$. It is also known that these finite band gap potentials are solutions of higher order Korteweg–deVries (KdV)-type nonlinear evolution equations [19].

2.2. Symmetries

At this stage, it is worth pointing out the symmetries of the GAL potential (3) and hence the corresponding Schrödinger equation (4).

1. The potential (3) and the Schrödinger equation (4) remain unchanged when any one (or more) of the four parameters a, b, f, g changes to $-a - 1, -b - 1, -f - 1, -g - 1$, respectively.
2. Under the translation $x \to x + K(m)$, the GAL potential $[a, b, f, g]$ goes to the potential $[b, a, g, f]$. Hence, both GAL potentials must have the same energy eigenvalues and the corresponding energy eigenfunctions are simply related [4]:
$$\begin{aligned} &E(b, a, g, f; m) = E(a, b, f, g; m), \\ &\psi(x, b, a, g, f; m) \propto \psi(x + K(m), a, b, f, g; m). \end{aligned} \quad (6)$$
3. Similarly, by considering the translations $x \to x + K(m) + \mathrm{i}K'(m)$ and $x \to x + \mathrm{i}K'(m)$, it is easy to show that
$$\begin{aligned} &E(f, g, a, b; m) = E(a, b, f, g; m), \\ &\psi(x, f, g, a, b; m) \propto \psi(x + K(m) + \mathrm{i}K'(m), a, b, f, g; m). \end{aligned} \quad (7)$$
$$\begin{aligned} &E(g, f, b, a; m) = E(a, b, f, g; m), \\ &\psi(x, g, f, b, a; m) \propto \psi(x + \mathrm{i}K'(m), a, b, f, g; m). \end{aligned} \quad (8)$$

Thus, once we obtain the eigenvalues and eigenfunctions of a given GAL potential $[a, b, f, g]$, then we immediately know the eigenvalues and eigenfunctions of three other potentials: $[b, a, g, f]$, $[f, g, a, b]$ and $[g, f, b, a]$. Note that relations (6)–(8) all involve the same modulus parameter m.

We shall now derive three duality relations which involve a change of modulus parameter.

2.3. Duality relation I

We start from the Schrödinger equation (4) for the GAL potential (3). On using the relations [9]

$$\begin{aligned} \sqrt{m}\,\mathrm{sn}(x, m) &= -\mathrm{dn}[\mathrm{i}x + K'(m) + \mathrm{i}K(m), 1 - m], \\ \mathrm{dn}(x, m) &= \sqrt{1 - m}\,\mathrm{sn}[\mathrm{i}x + K'(m) + \mathrm{i}K(m), 1 - m], \\ \sqrt{m}\,\mathrm{cn}(x, m) &= \mathrm{i}\sqrt{1 - m}\,\mathrm{cn}[\mathrm{i}x + K'(m) + \mathrm{i}K(m), 1 - m], \end{aligned} \quad (9)$$

and defining a new variable $z = \mathrm{i}y + K'(m) + \mathrm{i}K(m)$, the Schrödinger equation (4) takes the form

$$-\frac{\mathrm{d}^2}{\mathrm{d}z^2}\psi(z) - \left[a(a+1)(1-m)\mathrm{sn}^2(z,1-m) + g(g+1)(1-m)\frac{\mathrm{cn}^2(z,1-m)}{\mathrm{dn}^2(z,1-m)} + f(f+1)\frac{\mathrm{dn}^2(z,1-m)}{\mathrm{cn}^2(z,1-m)} + b(b+1)\frac{1}{\mathrm{sn}^2(z,1-m)}\right]\psi(z) = -[a(a+1)+b(b+1)+f(f+1)+g(g+1)+E]\psi(z). \tag{10}$$

On comparing equations (4) and (10), we then have the first remarkable duality relations [4]

$$\begin{aligned} &E(a,b,f,g;m) = [a(a+1)+b(b+1)+f(f+1)+g(g+1)] - E(a,g,f,b;1-m),\\ &\psi(x,a,b,f,g;m) \propto \psi(\mathrm{i}x + K'(m) + \mathrm{i}K(m), a, g, f, b; 1-m), \end{aligned} \tag{11}$$

which are valid for the energy states corresponding to either the band edges or mid-band states. Note that here, a, b, f, g can be arbitrary real numbers and are not restricted to integer values.

2.4. Duality relation II

Using the formulae [9]

$$\begin{aligned} &\mathrm{sn}\left(\sqrt{m}x, \frac{1}{m}\right) = \sqrt{m}\,\mathrm{sn}(x,m), \qquad \mathrm{cn}\left(\sqrt{m}x, \frac{1}{m}\right) = \mathrm{dn}(x,m),\\ &\mathrm{dn}\left(\sqrt{m}x, \frac{1}{m}\right) = \mathrm{cn}(x,m) \end{aligned} \tag{12}$$

and redefining a new variable $z = \sqrt{m}y$, the Schrödinger equation (4) takes the form

$$-\frac{\mathrm{d}^2}{\mathrm{d}z^2}\psi(z) + \left[\frac{a(a+1)}{m}\mathrm{sn}^2\left(z, \frac{1}{m}\right) + \frac{f(f+1)}{m}\frac{\mathrm{cn}^2\left(z,\frac{1}{m}\right)}{\mathrm{dn}^2\left(z,\frac{1}{m}\right)} + b(b+1)\frac{\mathrm{dn}^2\left(z,\frac{1}{m}\right)}{\mathrm{cn}^2\left(z,\frac{1}{m}\right)} + g(g+1)\frac{1}{\mathrm{sn}^2\left(z,\frac{1}{m}\right)}\right]\psi(z) = \frac{E(m)}{m}\psi(z). \tag{13}$$

On comparing equations (4) and (13), we then have the second duality relation [20]

$$\begin{aligned} &E(a,b,f,g;m) = mE\left(a,f,b,g,\frac{1}{m}\right),\\ &\psi(x,a,b,f,g;m) \propto \psi\left(\sqrt{m}x, a, f, b, g; \frac{1}{m}\right). \end{aligned} \tag{14}$$

2.5. Duality relation III

We again start from the Schrödinger equation (4) and now use the formulae [9]

$$\begin{aligned} &\mathrm{sn}(x,m) = \frac{\mathrm{sn}[\sqrt{1-m}x, \tilde{m}]}{\sqrt{1-m}\,\mathrm{dn}[\sqrt{1-m}x, \tilde{m}]}, \qquad \mathrm{cn}(x,m) = \frac{\mathrm{cn}[\sqrt{1-m}x, \tilde{m}]}{\mathrm{dn}[\sqrt{1-m}x, \tilde{m}]},\\ &\mathrm{dn}(x,m) = \frac{1}{\mathrm{dn}[\sqrt{1-m}x, \tilde{m}]}, \qquad \tilde{m} = \frac{-m}{1-m}, \end{aligned} \tag{15}$$

On defining a new variable $z = \sqrt{1-m}\,y$, the Schrödinger equation (4) takes the form

$$-\frac{\mathrm{d}^2}{\mathrm{d}z^2}\psi(z) + \left[-\frac{mb(b+1)}{1-m}\mathrm{sn}^2(z,\tilde{m}) - \frac{ma(a+1)}{1-m}\frac{\mathrm{cn}^2(z,\tilde{m})}{\mathrm{dn}^2(z,\tilde{m})} + f(f+1)\frac{\mathrm{dn}^2(z,\tilde{m})}{\mathrm{cn}^2(z,\tilde{m})} + g(g+1)\frac{1}{\mathrm{sn}^2(z,\tilde{m})}\right]\psi(z) = \frac{1}{1-m}E\psi(z) - \frac{m}{1-m}[a(a+1)+b(b+1)+f(f+1)+g(g+1)]\psi(z). \quad (16)$$

On comparing equations (4) and (16), one gets the third duality relation [20]

$$E(a,b,f,g;m) = (1-m)E(b,a,f,g;\tilde{m}) + m[a(a+1)+b(b+1)+f(f+1)+g(g+1)],$$
$$\psi(x,a,b,f,g;m) \propto \psi(\sqrt{1-m}x,b,a,f,g;\tilde{m}). \quad (17)$$

Using the three duality relations as given by equations (11), (14) and (17) and the three symmetry relations (6)–(8), it is easily shown that once the eigenstates of a given potential are known, then we can immediately obtain the energy eigenstates of all the 24 potentials, obtained by permuting the four parameters a, b, f, g. Hence, out of these 24 potentials, there is only one independent potential.

2.6. QES solutions

Let us now seek solutions of the Schrödinger equation (4) for the PT-invariant GAL potential (3). On making the ansatz

$$\psi(x) = \mathrm{dn}^{-b}(y)\mathrm{sn}^{-g}(y,m)\mathrm{cn}^{-f}(y,m)\phi(y), \qquad y = x - \frac{K(m)}{2} - \frac{\mathrm{i}K'(m)}{2}, \quad (18)$$

it is easily shown that ϕ satisfies the equation [4]

$$\phi''(y) + 2\left[mb\frac{\mathrm{sn}(y,m)\,\mathrm{cn}(y,m)}{\mathrm{dn}(y,m)} - g\frac{\mathrm{cn}(y,m)\,\mathrm{dn}(y,m)}{\mathrm{sn}(y,m)} + f\frac{\mathrm{dn}(y,m)\,\mathrm{sn}(y,m)}{\mathrm{cn}(y,m)}\right]\phi'(y) + [Qm\,\mathrm{sn}^2(y,m) - R]\phi(y) = 0, \quad (19)$$

where

$$Q = (b+g+f)(b+g+f-1) - a(a+1), \qquad R = -E + (f+g)^2 + m(g+b)^2. \quad (20)$$

It is well known [21] that this is a quasi-exactly solvable (QES) problem and one obtains $n+1$ solutions when $a+b+f+g$ (or any other combination obtained by interchanging one or more of the four parameters a, b, f, g with $-a-1, -b-1, -f-1, -g-1$, respectively) is equal to $2n$ with $n = 0, 1, 2, \ldots$. In particular, for $n = 0$, the corresponding QES energy eigenstate is given by [4]

$$E = (a+b)^2 + m(b+g)^2, \qquad \phi = \text{constant}. \quad (21)$$

For higher values of n, the eigenvalues can be obtained by solving an equation of order $(n+1)$, while the corresponding eigenfunctions are of the form

$$\phi = \sum_{k=0}^{n} A_k\,\mathrm{sn}^{2k}(y,m). \quad (22)$$

One can in fact precisely specify the nature of the band-edge eigenfunctions for any GAL potential $[a, b, f, g]$ when the four parameters take integer values [20].

2.7. Supersymmetry, PT-symmetry and partner potentials

One can now start with the ground-state as well as the excited-state eigenfunctions of various GAL potentials discussed above and using supersymmetry obtain the corresponding SUSY partner potentials. This procedure leads to several new periodic potentials with a finite number of band gaps. As emphasized in the introduction, unlike the real potentials, if we start with a complex PT-invariant potential, then one can use any excited-state wavefunction to calculate the corresponding superpotential W.

For example, consider the Lamé potential $a(a+1)m\,\mathrm{sn}^2(y,m)$, which for integer a has $2a+1$ band edges. One could start from any of the $2a+1$ eigenfunctions and obtain the corresponding partner potential having the same band edges. Further from the known band-edge eigenfunctions of these $(2a+1)$ strict isospectral potentials, we can start from any of the remaining $2a$ eigenfunctions and construct still different partner potentials with the same band edges. In the same manner, we can construct many potentials which will be isospectral to the PT-invariant GAL potentials (3). In particular, starting from any of the $n+1$ eigenfunctions as given by equations (22) and (19), it follows that (for any $n > 0$) the non-GAL partner potentials of the form

$$V_+(x) = V_{\mathrm{GAL}}(x) - 2\frac{\mathrm{d}^2}{\mathrm{d}x^2}\ln\psi_{\mathrm{GAL}}(x) \tag{23}$$

are also finite gap potentials.

Let us enquire as to which of the GAL potentials are isospectral to each other. For example, consider the Lamé potential $6m\,\mathrm{sn}^2(y,m)$ (i.e., $[2,0,0,0]$) which has five band edges. If we start from one of its excited-state eigenfunction, say $\mathrm{cn}(y,m)\,\mathrm{dn}(y,m)$, then it is easy to show that the corresponding partner potential V_+ is another GAL potential, i.e. $[1,1,1,0]$. By examining several such well-known Lamé and AL potentials, we had conjectured [4] that for integer values of a, b, Lamé and AL potentials have the same band-edge eigenvalues as some GAL potentials, the explicit relationship being

$$\begin{aligned}
[a,0,0,0] &\equiv \left[\frac{a}{2},\frac{a}{2},\frac{a}{2},\frac{a-2}{2}\right], \qquad a = \text{even integer},\\
[a,0,0,0] &\equiv \left[\frac{a+1}{2},\frac{a-1}{2},\frac{a-1}{2},\frac{a-1}{2}\right], \qquad a = \text{odd integer},\\
[a,b,0,0] &\equiv \left[\frac{a+b}{2},\frac{a+b}{2},\frac{a-b}{2},\frac{a-b-2}{2}\right], \qquad a+b = \text{even integer},\\
[a,b,0,0] &\equiv \left[\frac{a+b+1}{2},\frac{a+b-1}{2},\frac{a-b-1}{2},\frac{a-b-1}{2}\right], \qquad a+b = \text{odd integer}.
\end{aligned} \tag{24}$$

Note that we are using the notation '$\equiv$' to denote 'same band-edge eigenvalues', but not identical potentials. It is interesting to note that recently Takemura [22] has verified the conjectures expressed in (24). Further, more generally, he has proved that the GAL potential $[a,b,f,g]$ for integers a, b, f, g has the same band-edge eigenvalues as another GAL potential, with the explicit relationship depending on whether $N \equiv a+b+f+g$ is an even or an odd integer. If N is an even integer, the relationship is

$$[a,b,f,g] \equiv \left[\frac{a+b+f-g}{2},\frac{a+b+g-f}{2},\frac{a+f+g-b}{2},\frac{b+f+g-a}{2}\right], \tag{25}$$

while if N is an odd integer, then the relationship is

$$[a,b,f,g] \equiv \left[\frac{a+b+f+g+1}{2},\frac{a+b-f-g-1}{2},\frac{a+f-b-g-1}{2},\frac{a+g-b-f-1}{2}\right]. \tag{26}$$

He has also shown that if a, b, f, g are all half-integers and their sum is an even integer, then the band-edge energy eigenvalues of the potential $[a, b, f, g]$ are the same as two other GAL potentials where also all four parameters are half-integers with their sum being an even integer. In particular, the explicit relationships are

$$\begin{aligned}\left[a = k + \frac{1}{2}, b = l + \frac{1}{2}, f = n + \frac{1}{2}, g = p + \frac{1}{2}\right] \\ \equiv \left[\frac{k+l+n+p+3}{2}, \frac{k+l-n-p-1}{2}, \frac{k+n-l-p-1}{2}, \frac{k+p-n-l-1}{2}\right] \\ \equiv \left[\frac{k+l+n-p+1}{2}, \frac{k+l+p-n+1}{2}, \frac{k+n+p-l+1}{2}, \frac{l+n+p-k+1}{2}\right]. \quad (27)\end{aligned}$$

Recently, we [20] have further generalized Takemura's results and have shown that any GAL potential $[a, b, f, g]$ has the same QES energy eigenvalues as two other potentials as given by equations (25) and (26) provided that one of the following requirements holds: (i) a, b, f, g are all integers or (ii) at least one or more of the four parameters a, b, f, g are half integral while others could take any arbitrary value or (iii) a, b, f, g are any arbitrary numbers but either $a + b + f + g$ or any other combination obtained by replacing one or more of these numbers by $-a - 1, -b - 1, -f - 1, -g - 1$, respectively, is an even integer. We might add here that in case a, b, f, g are all integers, it is easily checked from equations (25) and (26) that the third partner is the potential where are all four parameters are half integral with their sum being an odd integer. For example, we predict that the QES mid-band energy eigenvalues of the potential $[3/2, 1/2, 1/2, 1/2]$ are precisely the same as the band-edge eigenvalues of the Lamé potential $6m\,\mathrm{sn}^2(y, m)$.

2.8. Independent potentials with a band gaps [a, b, f, g = integers]

Given a potential of the form $[a, b, f, g]$ with $a \geqslant b \geqslant f \geqslant g \geqslant 0$, we now discuss how many independent GAL potentials there are with exactly a band gaps.

From equation (5) for the number of band gaps, it is obvious that the potentials $[a, b, f, g]$ are of two types, those with a band gaps and those with more than a band gaps. Further, amongst the potentials with a band gaps, several potentials are not independent being related by equations (25) and (26). On making a careful count of all these facts, we have shown [20] that the number of independent GAL potentials $[a, b, f, g]$ with a band gaps is given by

$$\begin{aligned} N_a &= \frac{1}{18}[a^3 + 9a^2 + 6a + 2], & a &= 1 (\mathrm{mod}\, 3) \\ N_a &= \frac{1}{18}[a^3 + 9a^2 + 6a - 2], & a &= 2 (\mathrm{mod}\, 3) \\ N_a &= \frac{a}{18}[a^2 + 9a + 6], & a &= 0 (\mathrm{mod}\, 3). \end{aligned} \quad (28)$$

Let us make several observations related to the above-given results.

(1) From equations (28), we see that the number of independent potentials N_a with a band gaps increases as a increases. In fact, the number of independent potentials having $a = 1, 2, 3, 4, 5, \ldots$ band gaps is $N_a = 1, 3, 7, 13, 21, \ldots$. This implies [19] that while there is only one independent KdV equation of third order, there should be three KdV-type equations of fifth order, seven such equations of seventh order, etc.

(2) The three distinct potentials with two band gaps are $[2, 0, 0, 0]$, $[2, 1, 0, 0]$, $[2, 1, 1, 0]$. The band structure of the first two potentials has already been studied [8, 10]. It turns out [20] that the potential $[2, 1, 1, 0]$ has four band edges of period $4K$ and one band

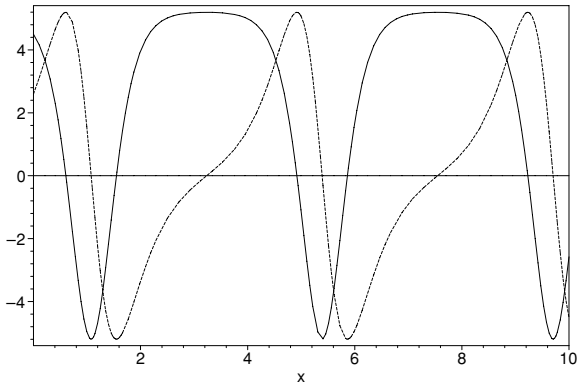

Figure 1. Real and imaginary parts of the potential [2, 0, 0, 0].

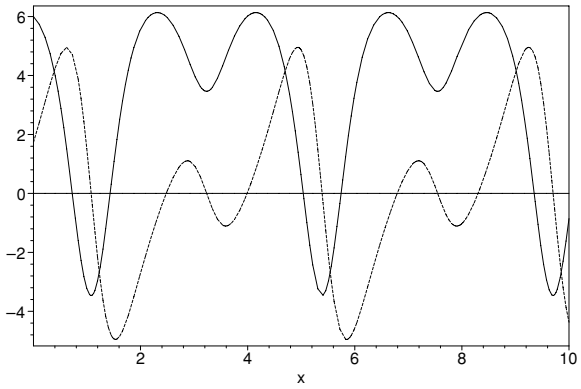

Figure 2. Real and imaginary parts of the potential [2, 1, 0, 0].

edge of period $2K$. To get a visual feel for the complex potentials, we have given plots of the three potentials in figures 1–3. The real and imaginary parts of all potentials are given by the solid and dashed lines, respectively, and we have chosen the value of the modulus parameter to be $m = 0.75$. Note that since we are using the argument $x - K(m)/2 - \mathrm{i}K'(m)/2$, there are no singularities on the real axis.

(3) In case $a + b + f + g$ is an even integer and further if $a + g = b + f$, then both sides of equation (25) are in fact identical, i.e. in that case the corresponding potential is self-dual. Similarly, when $a + b + f + g$ is an odd integer and if further $a = b + f + g + 1$, then both sides of equation(26) are identical, i.e. it is a self-dual potential.

(4) Clearly, all potentials with $a + g > b + f$ or $a > b + f + g + 1$, depending on if $a + b + f + g$ is an even or an odd integer, have partner potentials as given by equations (25) or (26), respectively. Now out of these, some are SUSY partner potentials while the rest are merely partner potentials. Now if two GAL potentials are SUSY partners, then one of their eigenfunction must be related to each other by $\psi_{\mathrm{II}} = \psi_{\mathrm{I}}^{-1}$. Further, these two eigenfunctions must have the form

$$\mathrm{dn}^{\alpha}(x, m)\,\mathrm{cn}^{\beta}(x, m)\,\mathrm{sn}^{\gamma}(x, m). \tag{29}$$

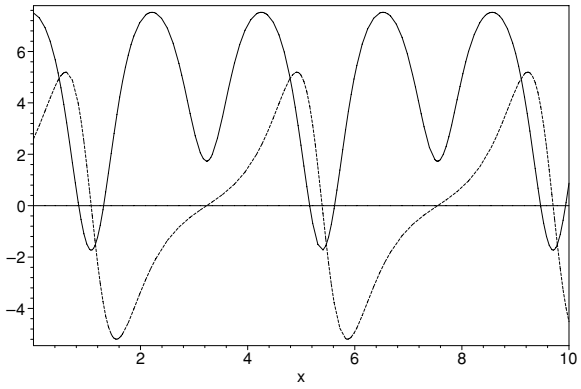

Figure 3. Real and imaginary parts of the potential [2, 1, 1, 0].

Besides, from [4], we know that if the QES eigenfunction is of the form (29), then $b = -\alpha$, $f = -\beta$, $g = -\gamma$ and $a + b + f + g = 0$. From here it is easy to show that the potential $[a, b, f, g]$ with a band gaps has a SUSY GAL partner provided either $a + g = b + f + 2$ or $a = b + f + g + 3$ depending on if $a + b + f + g$ is an even or an odd integer, respectively. Hence, all potentials of the form $[a, b, f, g]$ with a band gaps and satisfying $a + g > b + f + 2$ or $a > b + g + f + 3$ have merely partner potentials of the form (25) or (26), respectively. It is worth emphasizing that while these partner potentials have the same band-edge eigenvalues but none of them are SUSY partner potentials. For example, [2, 0, 0, 0] and [1, 1, 1, 0] are SUSY partner GAL potentials with two band gaps. Similarly, [3, 1, 0, 0] and [2, 2, 1, 0] are SUSY partner GAL potentials with three band gaps. On the other hand, [4, 0, 0, 0] and [2, 2, 2, 1] are merely partner potentials with four band gaps.

In this paper, we have discussed GAL potentials when all four parameters a, b, f, g are integers and there are a finite number of band gaps. For completeness, we mention that several results are also known for potentials in which at least one of the parameters is a half-integer and one has an infinite number of band gaps. In particular, relations (25) and (26) are still simultaneously valid and one can obtain the QES energy eigenvalues. Further, using these QES energy eigenvalues and the well-known connection [21] between the GAL potential problem and Heun's equation [23], it is possible to obtain the QES energy eigenfunctions for the GAL potentials $[a, 1/2, f, g]$, $[a, b, 1/2, g]$, $[a, b, f, 1/2]$ and interesting connections between the different solutions of Heun's equation. These and other related results are discussed in [20].

References

[1] Bender C M and Boettcher S 1998 *Phys. Rev. Lett.* **80** 5243
For a recent review of this field see Bender C M, Brody D C and Jones H F 2003 *Am. J. Phys.* **71** 1095 and references therein
[2] Khare A and Mandal B P 2000 *Phys. Lett.* A **272** 53
Ahmad Z 2001 *Phys. Lett.* A **286** 231
Cervero J M and Rodriguez A 2004 *J. Phys. A: Math. Gen.* **37** 10167
[3] Khare A and Sukhatme U P 2004 *Phys. Lett.* A **324** 406
[4] Khare A and Sukhatme U P 2005 *J. Math. Phys.* **38** 082106
[5] Magnus W and Winkler S 1966 *Hill's Equation* (New York: Wiley)
[6] Arscott F M 1981 *Periodic Differential Equations* (Oxford: Pergamon)

[7] Whittaker E T and Watson G N 1980 *A Course of Modern Analysis* (Cambridge: Cambridge University Press)
[8] Khare A and Sukhatme U P 1999 *J. Math. Phys.* **40** 5473
[9] For the properties of Jacobi elliptic functions, see for example Abramowitz M and Stegun I 1964 *Handbook of Mathematical Functions* (New York: Dover)
Gradshteyn I S and Ryzhik I M 2000 *Table of Integrals, Series and Products* (New York: Academic)
[10] Khare A and Sukhatme U P 2001 *J. Math. Phys.* **42** 5652
[11] Khare A and Sukhatme U P 2004 *J. Phys. A: Math. Gen.* **37** 10037
[12] Khare A 2001 *Phys. Lett.* A **288** 69
Ganguli A 2002 *J. Math. Phys.* **43** 5310
[13] See for Example Cooper F, Khare A and Sukhatme U P 2001 *Supersymmetry in Quantum Mechanics* (Singapore: World Scientific) and detailed references contained therein
[14] Sinha A and Roy P 2003 *Preprint* quant-ph/0312085
[15] Treibich A and Verdier J-L 1990 *C. R. Acad. Sci., Paris* I **311** 51
Treibich A and Verdier J-L 1992 *Duke Math. J.* **68** 217
[16] Darboux G 1882 C R t XCIV **25** 1645
[17] de Sparre C 1883 *Acta Math.* **3** 105, 289
[18] Gesztesy F and Weikard R 1995 *Math. Z.* **219** 451
Treibich A and Verdier J-L 1997 *C. R. Acad. Sci. Paris* I **325** 1101
[19] Novikov S P 1974 *Funct. Anal. Prilozen* **8** 54
[20] Khare A and Sukhatme U 2006 *J. Math. Phys.* **47** at press (*Preprint* quant-ph/0602105)
[21] Brihaye Y, Giller S and Kosinski P 1995 *J. Phys. A: Math. Gen.* **28** 421
[22] Takemura K 2005 *Preprint* math.CA/0508093
[23] For an up to date mathematical summary of Heun's equation, see Ronveaux A (ed) 1995 *Heun's Differential Equation* (Oxford: Oxford University Press)